SOME SUGGESTIONS IN THE TEACHING OF GEOMETRY.

By Isaac J. Schwatt.

The average student seems to have less understanding of geometry than of any other of the elementary mathematical disciplines. Even calculus—of which Luebsen, one of the ablest writers of mathematical text-books, has said, that out of ten who study the subject only two understand it and still fewer are able to apply it—is not excepted. This is in part due to the extreme difficulty of the subject of geometry, the age at which the study of it is usually attempted, and to the relatively small amount of time alloted to it.

Instruction in geometry ordinarily consists in the study of the theorems and the proofs given in the text with comparatively little work in the solution of problems of construction. In this respect instruction in geometry differs from that in algebra. The latter consists mostly in the working of examples and problems; very little theory is given. One need only compare the examination questions in these two subjects set by the schools and for admission to colleges.

Under the present system of marking, it is possible for a candidate to pass almost any of the examination papers in geometry which I have seen, if he is able to prove the theorems as given in most text-books, without having acquired facility in solving constructions. I know of a case where the insertion in the entrance examination paper in algebra for admission to a university of one or two questions pertaining to the theory of the subject, brought serious protest from teachers who prepare students for that institution. To follow the general trend and expediency, it was deemed advisable to omit such questions in subsequent papers.

While a pupil may pass an examination in geometry by memorizing theorems and proofs, to pass the examination in algebra he must understand its processes and acquire facility in their use. Even if the examination paper in algebra does not contain any

questions on the theory of the subject, the data and the figures of the examples on it are different from any he has seen before. The working of constructions develops in the mind of the learner just such faculties and such powers as he will be in need of whatever his vocation or occupation in life.

One of the first requisites for success in life, whether in the performance of professional duties or in commercial enterprise, consists in the ability to bring all the possibilities of a situation instantaneously before the mind and to select the one which will give the required result. Take for instance the work of the physician. From his knowledge of the human body in its healthy state and from his experience with it in a pathological state, the physician must make combinations and apply them to the case to be diagnosed in such a manner as will enable him to decide wherein the deficiency of the human mechanism lies. Exactly such training is furnished by the working of constructions. The work in constructions calls for continuous concentration and uninterrupted attention, for the utmost patience, for the quality of not being discouraged by failure, and for persistency and tenacity to continue until the desired result has been accomplished.

In this work more than in any other mathematical discipline the student depends on the power of mind which he has acquired—not alone by the study of mathematical subjects, but also as a result of his whole intellectual training. But it takes a greater maturity of mind than is attained by the average high school pupil to derive these mental benefits from the study of geometry, and to acquire a thorough knowledge of geometrical concepts and facility in solving constructions. This remains true even if an undue amount of time and energy is expended, both by the pupil and teacher, on the study of the subject.

One who has never studied algebra, who has never heard of equations, and who has not even had a systematic training in arithmetic, may yet be able to solve some of the most difficult problems of algebra. But I doubt whether any one, however able, can by observation alone and without any systematic knowledge of the principles of geometry, arrive at some of the properties and proofs, of the triangle, say.

The steps of a geometrical proof should follow each other

in a strictly logical sequence. This, however, is not always the case. Many proofs require a number of auxiliary lines which to the beginner appear more or less artificial. In many others the successive steps are not always connected so as to enable the student to deduce each step from the preceding by purely logical reasoning and with the aid of principles previously derived. In reproducing such proofs the learner will depend also on his memory. It is an interesting question as to how much the memory has to do with the derivation of mathematical principles and laws. None of us can remember all the proofs we have studied, nor is it necessary that we should. An examination in geometry ought therefore to be conducted in such a manner as to convince the examiner that the student has obtained the mental benefit which the study of the subject is capable of yielding. Such an examination should consist of constructions and proofs of theorems commonly called original exercises. The teacher will soon learn from the manner in which the student attacks these, the strength of his geometrical development.

The growth of the mental powers, the development of the mind, is a matter of evolution. The degree of difficulty of an idea, the amount of meditation and reflection necessary to make it entirely clear to the student, must be adapted not only to his natural ability, but also to his age.

To derive the fullest mental benefit from the study of formal geometry and especially from practice in working constructions, the subject should be taken up in college. The study of geometry is, in my opinion, more difficult than the ordinary course in logic, which, as a rule, is reserved for the upper classes in college.

There seems to be no provision in either the secondary school or the college, for the student to get any idea of the possibility of a space of higher dimensions than the one known to us. The idea is extremely fascinating and is excellent material for meditation and reflection. A system in which we are led by logical deductions to the conclusion that a hollow sphere can be turned inside out without breaking its shell, in which there are five conics instead of four, and so on, is one well calculated to stimulate thought and meditation. If geometry were taught in college, where students are more mature such topics might be taken up. One of the purposes of education should be to give the

learner a thirst for knowledge and a desire for meditation and reflection. How much happiness is to be derived from the ability to meditate! The man who meditates does not feel lonesome, time is not heavy on his hands. He who enjoys meditation and reflection will not seek pleasure in diversions which he cannot afford, nor in those which undermine his health and maybe his morals.

Although geometry of all mathematical disciplines may best serve the purpose of education, yet on the whole there is not as much attention paid to the study of geometry as to the study of algebra. Not only is geometry the more difficult subject, but algebra appears to be the more practical. The solution of many problems is greatly facilitated by the use of the equation. By means of algebraical processes, interest tables, tables for discount, etc., have been computed.

On the other hand geometry has little practical application and its truths are of little interest in the ordinary pursuits of life. Its only purpose is to increase the power of mind of the learner, to develop in him the ability to draw logical inferences from accepted premises. If its study does not accomplish this purpose, it must be considered a failure, a waste of energy and time, not only retarding the development of character, but having a pernicious effect on it.

There is in general very little plane geometry used in the higher branches of mathematics, in calculus for example, or even in analytical geometry. In these subjects most operations are performed analytically. To this is due the greater importance which is attached to the study of algebra. I ought, however, to add that some of the problems of plane geometry which occur in engineering are of greater difficulty than the average pupil with but one year's instruction in the subject is able to solve. Problems relating to the division of plane figures, or such as arise in connection with the laying of curved railroad tracks, or of tracks leading to a turn-table, require a high development in geometrical methods and great facility in geometrical construction.

Geometry is generally acknowledged to be a strong educational medium. Its ability to develop power of mind is very seldom questioned. It is therefore the more surprising that a member of our own craft should give expression to the following:

"We had also fondly cherished the traditional belief that geometry is a tonic for all mental shortcomings (and we do not propose to retreat completely from this position) but we do not now dare to place as much confidence as heretofore in the theory that wits sharpened on geometry will therefore present a keen edge in all other phases of mental activity. We have seen so many good geometricians who seemed to possess small logical sense in other affairs and so many good logicians who have small knowledge and little facility in geometry, that we can no longer believe in the inevitable transference of power gained in the study of geometry to effective use in other lines." *

Casual intercourse with a person is not, as a rule, of such a nature as to enable us to judge of his mental powers. We are inclined to consider a person, who by constant practice is able to perform his duties without much effort, as possessing extraordinary ability.

The mind which is capable of deep thought and deep reflection does not always possess the superficial brightness often mistaken for mental power. The person who is able to speak of his special work most convincingly has not necessarily a logical mind.

It is a matter of my personal observation that mechanics and artisans who possess great skill, do not always have the power of mind to apply it in the most advantageous manner. It is one thing to build a house and another thing to furnish it. It is one thing to have skill and another thing to make that skill most efficient. Little as we know whether a person has been benefited by a certain study, we know still less whether he would have been more benefited by another.

There is at the present time much discussion about the new education, which I take to be utilitarian education. In spite of this tendency, the purpose of the school ought to be first and foremost, to develop the character and the mind of the learner. Unless the school can accomplish this purpose, it should not undertake other problems, however practical they may appear. The best inheritance we can leave our children is strong bodies and honest characters together with strong minds.

We cannot give our pupils directions for all the problems and conditions they may meet in life. Even in the professional

* THE MATHEMATICS TEACHER, July, 1909, p. 161.

schools we cannot prepare the student for all the difficulties he may meet in the practice of his profession. New conditions may arise which will leave him helpless, if he has not the power of mind to conquer and to solve them.

I am in this connection reminded of the following story: A little girl was instructed by her mother how to act at a party which she was to attend. The mother was careful to tell her what she should say when the cake was offered for the first time, and what she should say when it was passed the second time. At the party it happened that the cake was offered a third time. Not having been instructed what to say in this case, the girl recalled that her father so tempted had remarked, "Hell, I will have another piece."

A new education should be such as to prepare the student for the ever new and changing conditions that arise in the evolution of mankind. With every day our environment changes. Education to be adequate must prepare the student to keep pace with the progress of the world. That progress expresses itself in keener competition, in an enlarged field of activity, necessitating greater discrimination, greater resourcefulness, greater ability to exploit new enterprises, and to distribute the ever increasing products of manufacture and invention. Education should ever be directed towards increasing the power of mind of the individual to meet and cope with all kinds of activities in their ever-broadening aspects.

The inventions of the last decades disclose some of the wonderful possibilities of applying the forces of nature. The telephone, the flying machine, and the speed attained on land and sea are a few achievements of which we formerly never dreamt. We can only conjecture how much more is in store for those who have the necessary mental power. With all deference to the statement in the Bible that, "Thou shouldst gain thy bread by the sweat of thy brow" (Gen. 3, 19), I cannot help feeling that man was put on earth not merely to provide for all his needs by laboring from sunrise to sunset, but for the betterment of the world, for his own intellectual and spiritual improvement, for meditation and for proper enjoyment of life.

If we but discover the true character of the forces of nature, we may be able to supply our wants without the expenditure of much energy and time. I feel that the exploit-

ing of the natural forces by mechanical processes and devices is still in its infancy. I feel that forces of nature which will enable us to gain the comforts and necessities of life surround us in abundance. We must only learn to harness and apply them. What we need is keenness of perception, power of mind, power of discrimination and all of those qualities that education improves, if it does not create.

Education must give the learner that attitude of mind which distinguishes the educated person from the uneducated. It is not the number of facts a person knows, it is the attitude of his mind, and the manner in which he thinks and acts that distinguishes him from those who have not had the advantages of education.

Even a foreign language should be taught for the mental development which it is capable of affording. Unless the student devotes more time than is generally allowed in schools and colleges to the study of a foreign language, he will not acquire the knowledge necessary for its practical use. In fact, a foreign language can be acquired only by long and continued practice, or by living among those who speak it.

We cannot arrange the curriculum of the elementary school, the high school, or the college with a view to the needs of the few. We cannot teach mathematics in such manner as will enable the one girl out of ten millions to use it if she will become a surveyor or an engineer. All our teaching must be done with the single view to what is good for the majority—to what will inspire all the pupils and help all of them to true happiness and high ideals.

I feel that the mathematical subjects taught in the secondary schools, are, on the whole, selected and arranged with the view of satisfying the requirements of the colleges and of the engineering schools. Students of the latter must have a knowledge of all the elementary branches of mathematics if they are to follow the instruction in analytical geometry and calculus. But the average high school pupil is unable to thoroughly understand and assimilate all the mathematical subjects covered in his course in the time allotted to their study. He is as unable to acquire a thorough knowledge of geometry and to drive the mental benefit which it is capable of giving him, as the college student is unable to thoroughly understand, digest and assimilate

all the ideas in calculus as they are presented in some of the most widely used text-books, and to acquire facility in performing its operations. The teacher of mathematics, like the teacher of any other subject, must never lose sight of the fact that his subject is only a part of the curriculum, and that the pupil must divide his time between all the subjects of his course.

One cannot help viewing with apprehension the ever increasing size of the text-books in mathematics. This tends to foster the pernicious habit of superficiality and acts against all the true purposes of education. We mathematicians are not the only sinners in this regard. While I am writing this a relative of mine is studying from a text-book on a medical subject containing about 1,000 pages. I understand that three hours a week for two terms of one half year each is given to the subject. I feel that it would be quite impossible for a person of average ability occupied with the many other subjects of the modern medical school curriculum, to acquire even a superficial knowledge of such a work in the time allowed to it. It would serve the cause of education if the text-books in most subjects were greatly reduced in size and if the student were held to a thorough knowledge of each and every idea presented in the book.

In formulating a course of study in any subject the principal questions ought to be, how much ground can be covered by the average student in a thorough manner so that he may gain a perfectly clear idea of each and every point in the subject in the time given to it.

Even the more difficult parts of elementary algebra, like the reduction of fractions and the reduction of surds, now given in secondary schools, should be taken up in college where the student is more mature. I know how difficult it is to break with established traditions. It will take the united efforts of all of us to bring about a change from existing conditions, to reduce the amount of each of the mathematical branches taught in secondary schools and in colleges, and to arrange the curriculum so that at any point the student may be properly prepared and sufficiently mature to successfully take up the next subject.

The present tendency seems to be towards higher entrance requirements, which as I understand it, implies more ground to be

covered. This can in general only be accomplished at the expense of thoroughness.

We should increase the intensity and efficiency of a student's knowledge of mathematics, rather than extend the scope and quantity of the subject matter covered. One clear idea, one proof which the student thoroughly understands, in which he knows the reason for every step taken, and which is entirely his own mental possession, is of infinitely greater benefit to his mental development than if he were to go over a great many theorems in a perfunctory manner. Instead of developing thoroughness, patience, perseverance, independence in thought and action, such a course fosters those pernicious qualities which it is the duty of the teacher to correct and overcome.

In the rush of our daily work, and in the rush of all our activities, we are apt to forget the true purpose of education. If the foundation of a subject is thorough, not only will this be fulfilled but also the pupil will be equipped to continue the study of that subject.

I cannot help but think that we send children to school much too early, that a great deal of the work in mathematics which the average pupil does in the elementary school and in the high school is beyond his mental abilities. We have over and over again heard of children who did not commence to study until much beyond the age at which the average child begins to attend school, and who have nevertheless accomplished more than those children who have been confined in the school room from their very early childhood. We all know how hazy our ideas were on some of the subjects we studied at school. It was not always due to lack of application on our part, more often it was the immaturity of our minds and the state of our mental faculties, even though they were normal for our age. No idea should be submitted to a child until his age and mental development permit him to gain a perfectly clear understanding of the idea and to assimilate and retain it.

Four courses in geometry should be given. First, a preliminary or introductory course. In this the pupil should draw simple figures, including such as may be termed geometrical ornaments. Squares, hexagons and triangles are laid out so as to form a compact plane—a plane is paved, so to speak, with these figures. Those children who have had the advantage of

Froebel's ideas as given in properly conducted kindergartens have in the study of geometry a decided advantage over those who have not had such instruction. Froebel believed in the early training in geometrical *forms*.

In a second course, called observational, inventional, or intuitive geometry, the pupil's conceptions of space-forms are extended; the properties of some of the geometrical figures are deduced by mental and physical inspection, that is by observation and by actual measurment. In this course the student will accumulate geometrical ideas and extend his geometrical instinct. The training derived from such a course is very important for the development of certain faculties in the young, especially those of observing, of discovering, of going open-eyed through the world. These faculties should be trained early. A desirable faculty in a person whatsoever his occupation may be, is the ability to estimate with some degree of accuracy, lengths and distances. This should be developed in the course in inventional geometry. Lengths of objects and of lines running in different directions are estimated by the pupil, and then by actual measurement he convinces himself of the accuracy of his estimate.

A most useful exercise which ought also to be taken up in this course is drawing to scale. This is one of the very few applications which geometry has in practical life. All of us might have occasion to find the distance between two places from their relative positions on a map, or to determine the dimensions of a building from its plan.

In a third course—demonstrative or formal geometry—the fundamental conceptions, formal proofs of some of the theorems and some easier constructions should be included. In this course the pupil deduces the properties of figures from axioms and definitions by a sequence of logical reasoning. He verifies by proof the truth of some of the properties he has previously obtained by observation and actual measurement. In observational and in inventional geometry a correct and accurate figure is indispensable. On the other hand for the formal demonstration of a theorem a correct figure is not essential. In fact, geometry has been defined as the science of "correct reasoning and bad figures."

A fourth course comprises the proofs of the more difficult theorems and the solution of geometrical constructions. This course should be taken up when the student is more mature, in the higher classes of the college.

It is a question whether inventional geometry should precede or accompany the course in demonstrative geometry. That is, whether intuitive geometry should be taken up first and then demonstrative geometry, or whether a pupil should be made to discover by measurement, etc., the properties of a figure and then by proof verify his estimate. With the latter mode of procedure he will better understand what he is expected to prove and perhaps be better prepared to proceed with the demonstration. This method is to some extent used in algebra. From a particular example a principle is deduced, and then the principle is formally proved. In this, inductive and deductive methods go hand in hand.

There is and has been a great deal of dissatisfaction with the teaching of geometry. This dissatisfaction is concerned, however, more with the inadequacy of the system of Euclid and of some of the systems designed to replace the Euclidean, than with the teaching of the subject as it actually effects the student. It is to be hoped, however, that an improvement in the system will bring about an improvement in the student's knowledge of the subject.

The principal objections to Euclid lie in the unsatisfactory definitions, axioms and proofs. Yet the definitions and axioms which have been suggested to replace those of Euclid have also been found wanting.

A definition should enable us to construct a geometrical concept in a clear and unambiguous manner by the aid of such concepts only as have already been defined.

It should be possible to completely construct a geometrical concept from its definition without the aid of imagination or of an *a priori* knowledge of it. When I say without imagination I mean that the latter should not supplement any defective parts of the definition. It is a question whether it is possible to include in a definition all the properties of a geometrical concept so that it will not be necessary to supplement it with any *a priori* knowledge. Euclid defines a point as that of which no part can be taken. But there is

a sense in which it may be said that no part of a molecule or of an atom can be taken. Again a point is often defined as the extremity of a straight line. This definition tells us what a point is in connection with a straight line, but not what it is independent of any other geometrical concept. Euclid gives the following definition of a straight line: Any line which lies evenly between the points in itself is a straight line. The definition of a straight line as the shortest distance between two points requires the definition of distance. The definition of a distance is based in turn on the definition of a straight line. A straight line is also defined as an extension in one dimension, but this requires first the definition of dimension. A straight line is sometimes defined as the path of a point moving constantly in the same direction, but direction cannot be defined without the idea of a straight line. We say that two straight lines have the same direction if they both pass through the same infinitely distant point, but an infinitely distant point cannot be defined without the conception of a straight line. No satisfactory definition of a straight line has yet been given.

I think it was Tyndall who said that no one ever gained a clear notion of what a straight line is from its definition. We are born with the intuitive or preconceived conceptions of a point and a straight line, which we call *a priori* conceptions.

Neither has a satisfactory definition of space been given. In fact, it is impossible not only to define space, but even to describe it.

Euclid defines a plane as a surface which lies evenly between the straight lines in itself. Besides not being clear, this presupposes the definition of a surface, which is difficult to define. The same objection holds against the definition: A plane is a surface such, that if any two points in it are joined by a straight line, the line will lie wholly in the surface.

Another definition of a plane is: A plane is obtained if a point outside of a straight line is joined to every point on the line. This definition does not determine all the points on the plane. It does not include those points which lie on the line passing through the given point and parallel to the given line. But parallel lines cannot be defined without first defining a plane. Still another definition of a plane is: A plane is the locus of all

points which are equally distant from two fixed points. This definition makes the assumption of a third dimension necessary and also the conception of equal distances. The definition: "If a right angle is revolving about one of its sides, the second side describes a plane," has the same difficulty.

The Italian mathematician Peano has given a definition of a plane which is rather too difficult for the beginner. It is: If one of three given points is connected by a straight line to a point on the line joining the other two points, and if from any point in this connecting line straight lines are drawn to points on the straight lines joining the three given points, we obtain a plane.

Euclid defines a plane angle as the inclination of lines one to another; that is, of two lines in a plane, meeting each other and not lying in a straight line. This definition would exclude a straight angle or a zero angle. An angle is also defined as the difference in direction between two straight lines intersecting in a point. But direction is a qualitative and not a quantitative conception. It therefore cannot be increased or diminished. Direction ought also be taken with respect to a fixed line.

Euclid's definition of a right angle is an angle which is equal to its adjacent angle. This requires the definition of adjacent angles. It also requires the knowledge that when two adjacent angles are equal, each is a right angle.

Another geometrical concept which is difficult to define is symmetry. Nothing will make this as clear as a concrete illustration. A pair of gloves which are equal in all their parts, yet cannot be made to cover each other, since the parts are arranged in opposite directions, are symmetrical.

The question arises, how many definitions are necessary to build up a consistent geometry so that no idea shall be assumed without having previously been defined. Most text-books fail to define for instance the bisector of an angle, or when three points are on the same straight line, etc. Three points are on the same straight line when the distance between two of them is equal to the sum or the difference of their distances from the third, etc. We have as much difficulty in properly defining some geometrical concepts as physicists have in defining matter and force, psychologists, mind and perception, and economists,

money and value. We acquire a knowledge of these ideas by repeatedly using them.

It is an important question whether we can evolve any concept of space-forms from abstract descriptions. No definition or description of a cube will convey to the immature child as definite an idea of a cube as will the seeing of it. There is also the question whether the theorems of geometry are constructed from fundamental concepts only, or also by the aid of experience and observation. It would be rather difficult to deduce the theorem that the sum of the three angles of a triangle is equal to two right angles from the definitions of an angle and a triangle only. It is a question whether we can obtain the theorem, that a straight line cuts a circle in two points, from the definitions of straight line and circle, without the aid of observation. It is an interesting inquiry whether we ever follow in geometry pure reasoning, entirely separate from our experience.

A more serious defect in Euclid is the inadequacy of the axioms. The 11th axiom: "If two lines are cut by a third, and the sum of the interior angles on the same side of the cutting line is less than two right angles, the lines will meet on that side when sufficiently produced," has defied the ingenuity of the ablest mathematicians to make it dependent upon simpler truths, that is, to prove it. It is quite surprising to find that in Euclid this 11th axiom is the converse of the seventeenth proposition: In a triangle the sum of two angles is less than two right angles. There are a number of substitutes for the eleventh axiom. One of them is: Through a point only one straight line can be drawn parallel to a given straight line, which sounds as if it were in no less need of proof.

Lobatschevsky and Bolyai, assuming that two parallel straight lines do meet, or what is the same, that the sum of the angles of a triangle is less than two right angles, thus eliminating the eleventh axiom of Euclid, have constructed an entirely consistent geometry called non-Euclidean geometry.

It cannot be denied that non-Euclidean geometry would serve much better that Euclidean geometry to develop power of mind in the learner, and the faculty of adhering to a sequence of strict logical reasoning. Since the truths of the theorems of non-Euclidean geometry do not agree with our

experience of space, their proofs require much closer concentration of mind. In Euclidean geometry our reasoning is supported by our experience of the space about us, while the results of non-Euclidean geometry contradict our observation and experience.

Some of the ablest mathematicians, among them Poincaré, Hilbert, Peano, Pasch, and Worpitzky have devised systems of axioms. But no system which is consistent, necessary, and sufficient has yet been established. It is amazing how greatly mathematicians differ as to the number and kind of assumptions which they consider necessary to build up a consistent and complete geometry. In this connection the question arises whether there is a finite number of axioms, or whether the number of axioms is infinite, and what is the first, the simplest axiom? Veronese says that he should like to be shown how a complete and consistent geometry can be constructed from Poincaré's axioms.

Mathematics is in more than one way receiving credit to which it is not entitled. Our science is looked upon by the representatives of other sciences as a perfect one, as the one in which definitions are complete, clear and unambiguous. It is therefore the more to be regretted that as far as the fundamental principles of our science is concerned, it may never be satisfactorily settled.

Several attempts have been made to devise a system of geometry different from the one which has come to us from Euclid. Some writers have tried to make the axioms more complete. Others have included some of the concepts of modern geometry. Others again have tried to evade the parallel axiom by first proving that the sum of the three angles of a triangle is equal to two right angles. One of the best known of the latter efforts is that of Thibaut. The proof is correct, but it assumes the revolving of a straight line about a point through 360 degrees to be equivalent to the revolving of the line about three points making the sum of the revolutions equal to 360 degrees.

The methods of proof in Euclid show serious defects. In some of the Euclidean proofs it is often difficult to recognize the connection between the beginning of the proof and what is to be proved. Euclidean proofs lack homogeneity of sys-

tem, and a general method of procedure. The proofs of a set of theorems even when they are closely related, have as a rule very little in common. In Euclidean geometry a proof refers to a fixed position of the straight lines and points of a figure. As soon as they change their positions the proof changes too, and very often the entire method of proof. Yet they all impress us with their high degree of excellence and ingenuity.

Although many of the writers on the subject of geometry are full of praise for the advantages to be derived in applying the principles and ideas of modern geometry—as discovered by Poncelet, Moebius, von Staudt and Steiner—to the study of elementary plane geometry, I have so far failed to discern any of them. Euclid once told his King, Ptolomæus, who found the study of geometry difficult, that "there is no royal road for geometry" but the followers of the modern geometry seem to think that this royal road has been found. We hear and read a great deal about the principle of duality, which it is claimed unifies the theorems and proofs of geometry and brings system into its chaos. This principle has greatly facilitated the study of curves and surfaces, of theorems and problems referring to groups of circles (*e. g.*, Malfatti's problem and the problems of Apollonius), and to inscribed and circumscribed polygons. I have failed however to find that the modern ideas possess any advantage in the study of the elementary properties of the triangle and the circle and in the proofs of those theorems which form the foundation of a large number of constructions. In modern geometry we have a much larger number of ideas to master than is necessary to follow Euclid. Concepts like infinitely distant points, infinitely distant lines, which play such an important part in modern geometry, are difficult ideas for the beginner. These concepts are necessary for the higher branches, but should be omitted in an elementary course in plane geometry.

Most of us have studied these ideas when we were of mature age and with a knowledge of Euclid. It is hard to abstract from the knowledge which we possess. I personally doubt that such an exclusion of ideas is at all possible. The beginner with no knowledge of Euclidean geometry would, I think, find great difficulty in following some of the ideas of the modern geometry.

I would hesitate to use with beginners the text books on the subject, at least those which I have seen.

It is quite significant that those who advocate the new ideas of geometry use Euclidean methods in solving constructions. Some of the proofs in modern geometry are easier than Euclid's but they lack the vigor of the latter. There are, however, a few ideas in modern geometry which might be introduced to advantage in the study of Euclid. One is the idea of reciprocity, duality and inversion.

Two straight lines determine a point.

Two points determine a straight line.

Three straight lines in general do not pass through the same point.

Three points in general do not lie on the same straight line.

Through two points an infinitely large number of circles can be passed.

Two straight lines are tangent to an infinitely large number of circles.

n points of a plane, give $\frac{n(n-1)}{2}$ connecting straight lines, provided not more than two of them are on the same straight line.

n straight lines in a plane intersect in $\frac{n(n-1)}{2}$ points, provided not more than two of them pass through the same point.

Some German mathematicians who wish to see the teaching of geometry reformed are not satisfied merely to add to Euclid some of the theorems of the new geometry, as for instance, those for harmonic points and harmonic divisions, points of similarity, pole and polar. They contend that it is not the amount of material, but it is the method which characterizes the new geometry and which is distinct from that of Euclid. They wish the subject developed entirely along the modern lines.

In teaching geometry it must be carefully decided which of the multitude of theorems shall be proved. The proof of theorems like, "Every circle has one center only" and "All right angles are equal," etc., may be of greater mental benefit to the student, than the proof of theorems the truth of which is not self-evident. The problem is then to select those theorems whose

proofs will be of most benefit to the learner. A proof must be free from inconsistencies and from such illusory statements as would stamp it in ordinary language as a sham proof. Some theorems can be proved in several ways. It is important to select the one which offers the best training for the mind. There is good mathematics and bad mathematics, if such an expression be permissible. We may prove a theorem or solve a problem in a short and direct way, or in a long and tedious way.

Another important question is whether a proof should be based as much as possible on first principles or on a succession of propositions. The Pythagorean theorem, for instance, can be proved in many ways, depending on the number and kind of basic theorems which are used.

Whatever system is used, whatever method is adopted in the teaching of a subject, no matter how excellent the text-book, it is after all the teacher, his ability to impart knowledge, to instill in the young the love for knowledge and for truth, his personality and his character, which is the most decisive factor in any system of education.

The knowledge of a teacher must be thorough and broad if he is to be able to meet all the possible conditions which may arise in the teaching of his subject. A teacher of geometry should be acquainted with all its systems and all its methods of construction, with Steiner's methods of carrying out a geometrical construction by means of a straight edge and a fixed circle in the plane without the use of compasses. He should be familiar with Mascheroni's methods for performing constructions by means of compasses only and without a straight edge, a straight line being given by means of two points only. No fixed method for the teacher to follow can be given. Whether to use the heuristic method, or the inductive or deductive, or the genetic method, will depend on the average mental capacity of the students, and on the special conditions and circumstances which may influence their minds. The method suitable to a bright spring day might prove ill adapted to a cloudy, heavy, fall day. Here is where the ingenuity and the ability of the teacher make themselves felt.

The student must thoroughly understand the content of a theorem, the meaning of each condition, and what is required to be proved. It is a pernicious habit, and the teacher must

carefully guard the student from giving way to it, to read over a problem or a theorem, and then before thoroughly understanding its content and meaning, to attempt its solution or demonstration. To make sure that the student understands what he is to do, he must be made to repeat the theorem in a wording different from that of the book. If a person is unable to express a thought, it is as a rule an unmistakable sign that the thought is not entirely clear to him. A student's claim that he understands the idea but is not able to express it, is very often unfounded. If he does understand the idea he will find words to express it, just as the teacher who has entire command of his subject will always be able to impart it to the learner. My experience has been that most students have difficulty in understanding quantitative conditions. In a theorem like the following: "In a triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides," the learner should understand that if the angle A of the triangle ABC is bisected by a line AD, the point D divides the side BC into two parts such that if AB is $\frac{3}{4}$ of AC, then BD will be $\frac{3}{4}$ of DC.

Few students thoroughly understand the meaning and significance of theorems like—"A line drawn parallel to a side of a triangle divides the other sides proportionally."

The young find thinking a hard task. A pupil will undergo the labor of memorizing a proposition and its proof rather than follow the more difficult course of concentrating his mind and trying to understand the meaning of the theorem and the successive steps of its demonstration. If the teacher is uninquisitive and is satisfied with the work as put on the blackboard or paper by the student, the latter is fairly sure of a high mark on a memorized proof. It is the duty of the teacher to require of the student the reasons for every step and to convince himself whether he thoroughly understands each condition in the theorem and each point of the proof. The teacher should not take any knowledge of the student for granted. What appears to be self evident to us may be entirely obscure to the student. He may not even be conscious that the idea is not clear to him until he is questioned about it. A good plan is to prove a theorem from the given conditions, and then after accepting its truth, invert the order of reasoning and see whether the given conditions must

prevail. The student will then better understand the reasons of each step of the proof. He will also better appreciate the necessity for all the steps taken in the course of the proof.

The teacher ought to work before the class a number of constructions, showing in detail the methods by which their solutions are obtained. In this way each student will acquire a part of the teacher's power of mind and of his knowledge and ability. Unconsciously, he will imitate the methods of the teacher. It is therefore extremely important that the pupil be instructed by a person who not only possesses a thorough knowledge of his subject, but who is equipped as well with the best methods. But if a student is to acquire the faculty of independent thought and action in striking out for himself, if he is to get the mental strength necessary to overcome obstacles different from any he has ever met before, and if he is to adapt himself to new and changed conditions, he will have to perform such constructions for himself. To find a solution will not only give him power of mind but will give him the joy and satisfaction of accomplishing something by his own strength. Searching for a solution will give the student a better grasp on the principles of the subject.

The explanations given by the teacher in any subject, and especially in mathematics, must be explicit and full. In the course of the discussion of a topic, the most elementary idea, the easiest formula with which the student is supposed to be familiar and which is involved in the explanation, should be carefully restated and expounded if by so doing the student's understanding will be clarified. Even if the student is familiar with the ideas used in the explanation, it may take some time for him to recall them, unless his mastery of the ideas is such as come, only from constant practice and repeated application. A good plan is for the teacher to restate all the principles and ideas involved in a proof or in the solution of a problem before proceeding with the work.

In building a house we do not erect it to the first story window and then order the window frames, thus interrupting the work. We order all the parts necessary for the entire structure before we commence building and then use each part as we need it. We must never forget that we teachers acquire by constant repetition a high degree of proficiency in the ideas

associated with our work. We have our definitions, principles and formulæ, so to speak, at our fingers' ends, but the student, no matter how able, can use them only with more or less effort.

It is highly important that in the course of an explanation no gaps should occur in the student's understanding. If he is to follow the explanations of the teacher and at the same time supplement them with such ideas as the teacher uses without restating, the student has two functions to perform, where one would be a sufficient tax on his mind.

While the teacher must insist firmly and uncompromisingly that the student do his full duty, he must never fail to encourage and stimulate the earnest worker with a word of praise, nor to reprimand, punish, or criticise the student who shirks his work. There is no one of us who is not encouraged and stimulated by sincere and well-earned praise. There is something wrong with us when we no longer care for the opinion of our fellow men.

If the student is neglectful of his duties, it is not sufficient that the teacher reprimand him and then dismiss him from consideration, nor is it sufficient in addition, to report him to the proper authorities, or to his parents or guardian, as deficient. It is the duty of the teacher to keep after the student and by convincing arguments get him to do his duty.

In urging students to do their duty and in rebuking them, great tact must be exercised and great care must be taken that the pressure brought to bear shall not have an effect opposite to the one desired. The student, who on account of inability or poor health, or for any other reasonable cause, is prevented from doing his work, must not be rebuked. He needs all our help and sympathy, all our kindness and encouragement.

There is no doubt that our first impressions are the most lasting. We remember incidents of childhood and youth much longer than those of manhood. Because of the novelty of the experience, they arouse the curiosity and produce a stronger impression on the mind. It is therefore of the greatest importance that those of us who are entrusted with the education of the younger pupils should possess the highest qualifications. If a difference in quality in the teachers for different grades be permissible, or if it be impossible to procure for all grades and classes teachers of the highest attainments,

the younger pupils should have the teacher who possesses the highest qualities of character and mind and the strongest personality. What is true of the lower grades of the elementary school is also true of the lowest classes in the college.

We are just passing through an era wherein young men rule in most of the occupations and professions. While we see unmistakable signs that this era is passing, yet there are positions and occupations in which the young man with all the push and physical vigor of youth is preferable to the man more calm and less enduring. But the calm repose of character and mind, and the ability to apply them for the benefit of the young, come only with age. There are many occupations in which one must serve as an apprentice before reaching the position of independent master. Those who intend to teach should be made to serve as apprentice in the class room of an experienced teacher. The strictest supervision by principals and superintendents cannot take the place of this suggested course. There are many young men and women, who after graduating from school and college while preparing for one of the professions or more remunerative occupations, are taking positions as teachers. They first teach then they go out into the world to gather experience. But the reverse procedure should be one of the requirements of the teaching profession. Those who intend to teach should by experience and observation broaden their minds, and then begin teaching. They will then understand human nature better. They will know what is expected of the future citizen and be able to influence him towards it. The teacher who uses his position as a stepping stone to something more remunerative is in most cases doing an infinite harm to the pupils entrusted to his instruction. Not considering teaching as his life work, he will as a rule not concentrate his efforts on better fitting himself for his duties.

UNIVERSITY OF PENNSYLVANIA,
December, 1909.